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# A NOTE ON MICHAEL'S PROBLEM CONCERNING THE LINDELÖF PROPERTY IN THE CARTESIAN PRODUCTS

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ABSTRACT. In this note we present a sketch of a negative solution of the Michael's conjecture which says that if the product $Y \times X$ is Lindelöf for every hereditarily Lindelöf space $Y$, then $Y \times X^\omega$ is Lindelöf for every hereditarily Lindelöf space $Y$.

**Introduction.** It is known that if $Y$ is a hereditarily Lindelöf space and $X$ is a separable metric space or a Lindelöf which is, complete in the sense of Čech, or a scattered Lindelöf space, then the product $Y \times X^\omega$ is Lindelöf. The first theorem is due to E. Michael, the second one to Frolik (see **[F]**) and the third one to the author (see **[AL$_1$]**) (see also **[T]**). I think that these results were the motivation of Michael's conjecture which says that if $Y \times X$ is Lindelöf for every hereditarily Lindelöf space $Y$ then $Y \times X^\omega$ is Lindelöf for every hereditarily Lindelöf space $Y$. In this note we are going to present the sketch of the proof that the answer to the Michael's conjecture is negative. The complete proof will appear in **[AL$_3$]**.

Let us recall that a subset of the Cantor set is analytic if it is an image of the irrational numbers and a subset of the Cantor set is coanalytic if it is a complement of an analytic set.

**Examples.**

EXAMPLE 1. *There exists $Z$ such that for every natural number $n$ and for every hereditarily Lindelöf space $Y$ the product $Y \times Z^n$ is Lindelöf but $Z^\omega$ is not.*

EXAMPLE 2. *Let $M$ be a coanalytic subset of the Cantor set which is not a Borel set. Then there is a space $X$ such that, for every Lindelöf space $Y$ and every natural number $n$, the products $Y \times X^n$ and $X^\omega$ are Lindelöf but $M \times X^\omega$ is not.*

It is easy to see that in order to obtain Example 1, it is enough to put $Z = M \times X$, where $M$ and $X$ are from Example 2.

**Terminology and notation.** Our topological terminology follows **[E]**. Let us recall that $X$ is a $P$-space if every $G_\delta$-subset of $X$ is open. The symbol $N$ stands for the set of natural numbers and $D = \{0, 1\}$ for the two-point discrete space. Greek letters are used to denote ordinal numbers, in particular $\omega$ stands for the first infinite ordinal number and $\omega_1$ for the first uncountable ordinal number. The symbol $D^\omega$ stands for the Cantor set and $B(i_0, \ldots, i_n)$, where $i_k$ for $0 \leqslant k \leqslant n$ is an element of $D$, denotes the set $\{i_0\} \times \{i_1\} \times \cdots \times \{i_n\} \times D \times D \times \cdots$. If $\alpha$ is an ordinal

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number then we shall identify it with the set of ordinal numbers less than $\alpha$. If $A$ is a set then the symbol $|A|$ stands for the cardinality of $A$.

**Auxiliary lemmas.** In this section we are going to formulate some lemmas. Their proofs will appear in [**AL$_3$**].

LEMMA 1. *If $N_k = N \times \{k\}$ for $k \in D$, $h$: $N_0 \oplus N_1 \to D$ is a mapping such that $h(N_k) = k$, and $B$ is an analytic subset of the Cantor set, then there is a closed subset $B_1$ of $(N_0 \oplus N_1)^\omega$ such that $f = h^\omega | B_1$ is a mapping from $B_1$ onto $B$.*

Let us attach to every limit countable ordinal number $\alpha$ a monotonically increasing sequence $(\alpha(n))_{n=1}^\infty$ of nonlimit ordinal numbers which converges to $\alpha$ in the order topology of $\omega_1$. Let us put $A = \{a \in D^{\omega_1}: |a^{-1}(1)| < \omega\} \cup \{a_\alpha \in D^{\omega_1}: \alpha$ is a limit ordinal number less than $\omega_1\xi$ and $a_\alpha^{-1}(1) = \{\alpha(n) : n \in N\}$. The topology on $A$ is induced by the sets of the form $B(a, \beta) = \{b \in A : b | \beta + 1 = a | \beta + 1\}$.

LEMMA 2. *The sapce $A$ is Lindelöf.*

The proof of Lemma 2 appeared in [**P**] (see also [**AL$_3$**]).

LEMMA 3. *If $\beta$ is a countable ordinal not less than $\omega$, then there is one-to-one function $h_\beta$: $N \to \beta$ from $N$ onto $\beta$ such that for every limit ordinal number $\alpha$ not greater then $\beta$ there are subsequences of natural numbers $(n_k^\alpha)_{k=1}^\infty$ and $(s_k^\alpha)_{k=1}^\infty$ such that the following conditions are satisfied*:

(a) $h_\beta(n_k^\alpha) = \alpha(s_k^\alpha)$, *for $k \in N$,*

(b) *for every $k \in N$ and for every $i \leqslant n_k^\alpha$, if $h_\beta(i) < \alpha$ then $h_\beta(i) < \alpha(s_k^\alpha)$.*

The sequence $(\alpha(s_k^\alpha))_{k=1}^\infty$ is a subsequence of $(\alpha(n))_{n=1}^\infty$, where $(\alpha(n))_{n=1}^\infty$ was defined in connection with Lemma 2.

Write $B = D^\omega \setminus M$, with $M$ from Example 2,

$$p = (p(0),\dots,p(n-1)) \in (N_0 \oplus N_1)^n, \quad \text{for } n \in N,$$

and put

$$H(p) = f((\{p(0)\} \times \cdots \times \{p(n-1)\} \times (N_0 \oplus N_1) \times \cdots \times (N_0 \oplus N_1) \times \cdots) \cap B_1)$$

and $Z(p) = \overline{H(p)}^{D^\omega}$, where $f$ and $B$ are from Lemma 1 and the closure operation is taken with respect to the topology of the Cantor set.

LEMMA 4. *If $p \in (N_0 \oplus N_1)^\omega$ and $Z(p | n)$ is not empty, for $n \in N$, then*

$$\bigcap \{Z(p | n) : n \in N\} = f(p) \in B.$$

Lemma 4 follows from the fact that $B_1$ is a closed subset of $(N_0 \oplus N_1)^\omega$.

In the sequel we shall need the following notation. Let us put, for $n \in N$, $H_n = \{H(p) : p \in (N_0 \oplus N_1)^n$ and $H(p)$ is not empty$\}$, $P_n = \{p \in (N_0 \oplus N_1)^n$: $H(p) \in H_n\}$ and $F_n = \{Z(p) : p \in P_n$ and $Z(p) = \overline{H(p)}^{D^\omega}\}$.

**Construction of the spaces $X$ and $M$ from Example 2.** Write $\Lambda = \{\alpha_\lambda : \lambda \in \{-1\} \cup \omega_1\}$, where

$$\alpha_\lambda = \begin{cases} 0, & \text{if } \lambda = -1, \\ \alpha_\theta + \omega, & \text{if } \lambda = \theta + 1, \\ \sup\{\alpha_\theta : \theta < \lambda\}, & \text{if } \lambda \text{ is a limit ordinal.} \end{cases}$$

If $\lambda = \theta + 1$, then put $\alpha_\lambda(n) = \alpha_\theta + n$ for $n \in N$. If $\lambda$ is a limit ordinal then let us attach to its a monotonically increasing sequence $(\lambda(n))_{n=1}^{\infty}$ of nonlimit ordinal numbers converging to $\lambda$ in $\omega_1$ and put $\alpha_\lambda(n) = \alpha_{\lambda(n)} + 1$. Let us put $A' = \{a \in D^{\omega_1} : |a^{-1}(1)| < \omega\} \cup \{a_{\alpha_\lambda} \in D^{\omega_1} : \alpha_\lambda \in \Lambda \text{ and } a_{\alpha_\lambda}^{-1}(1) = \{\alpha_\lambda(n) : n \in N\}\}$. Now let us consider $A'$ with the topology induced by $G_\delta$-subsets of the Tychonov product $D^{\omega_1}$. Then, according to Lemma 2, $A'$ is a Lindelöf space. Put $X = A'_0 \oplus A'_1$, where $A'_0$ and $A'_1$ are copies of $A'$. Notice that $X$ is a Lindelöf $P$-space, so by Noble's theorem (see [N]), the product $X^\omega$ is Lindelöf. From the fact that $X$ is a Lindelöf $P$-space, it follows very easily that for every Lindelöf space $Y$ and $n \in N$ the product $Y \times X^n$ has the Lindelöf property.

The description of an uncountable subset $L$ of $M \times X^\omega$ without points of condensation is the difficult part of the construction of Example 2. If $m = (m(n))_{n=0}^{\infty}$ is an element of $D^\omega$, then put $X_m = \prod_{n=0}^{\infty} A'_{m(n)}$. Let us notice that

$$P = \bigcup \{(m) \times X_m : m \in M\}$$

is a closed subset of $M \times X^\omega$. It is enough to define an uncountable set $L = \{l_\lambda : \lambda \in \{-1\} \cup \omega_1\}$ where $l_\lambda = (m_\lambda, x_\lambda)$, $m_\lambda \in M$, $x_\lambda \in X_{m_\lambda}$ and $m_\lambda \neq m_\beta$, for $\lambda \neq \beta$, without points of condensation in $P$. The set $L$ will be defined by transfinite induction with respect to $\lambda \in \{-1\} \cup \omega_1$.

If $x \in X$ then there is $r(x) \in D$ such that $x \in A'_{r(x)}$. If $x = (x(0),\ldots,x(n-1)) \in X^n$ for $n \in N$, then $r(x) = (r(x(0)),\ldots,r(x(n-1)))$. The symbols $a^0_{\alpha_\lambda}$ and $a^1_{\alpha_\lambda}$, for $\lambda < \omega_1$, will denote elements of $A'_0$ and $A'_1$ respectively which correspond to $a_{\alpha_\lambda}$ of $A'$, and the set $B(x)$ for $x \in X^n$, stands for the subset of $D^\omega$ which is equal to $B(r(x(0)),\ldots,r(x(n-1)))$. If $0 < \beta < \omega_1$ then $A'_i \mid \beta$ denotes the projection of $A'_i$, for $i = 0, 1$, onto the first $\beta$-coordinates. Let us put $X \mid \beta = A'_0 \mid \beta \oplus A'_1 \mid \beta$. If $x \in X$ then $x \mid \beta$ stands for an element of $A'_{r(x)} \mid \beta$ which is the projection of $x$ onto the first $\beta$-coordinates. If $x = (x(0),\ldots,x(n-1)) \in X^n$, where $n \in N$, then $x \mid \beta = (x(0) \mid \beta,\ldots,x(n-1) \mid \beta)$ and $X^n \mid \beta = (X \mid \beta)^n$. If $n \in N$ and $x \in X^k$, for $k \geqslant n$, or $x \in X^\omega$ then $x \mid n = (x(0),\ldots,x(n-1))$. In order to simplify the notation we shall identify $a^k_{\alpha_\lambda}$ with $a^k_{\alpha_\lambda} \mid \alpha_\gamma$ for $\gamma \geqslant \lambda$ and $k \in D$. Notice that $(a^k_{\alpha_\lambda})^{-1}(1)$ is a subset of $\alpha_\lambda$. If $x \in X$ and $x \mid \alpha_\lambda = a^k_{\alpha_\lambda} \mid \alpha_\lambda$ then $x = a^k_{\alpha_\lambda}$.

In every step of induction we shall also define some conditions which will restrict our freedom of choice of the elements of $L$ in the consecutive steps of induction. In the sequel these conditions will be called restrictions. The restrictions defined in the steps preceding the $\gamma$-step will ensure that every point $(m, (x(n))_{n=1}^{\infty})$ of $M \times X^\omega$, where $m \in M$ and $x(n)$ is not an element of $\{a^0_{\alpha_\gamma}, a^1_{\alpha_\gamma}\}$ for $n < \omega$, will not be a point of condensation of $L$. Notice that from the last fact it will follow that $L$ will not have points of condensation in $M \times X^\omega$. Indeed, if $(m, x)$ belongs to $M \times X^\omega$, then there is $\gamma$ in $\omega_1$ such that $x(n)$ does not belong to $\{a^0_{\alpha_\gamma}, a^1_{\alpha_\gamma}\}$ for $n$ less than $\omega$, so $(m, x)$ will not be a point of condensation of $L$. The role of restrictions will be played by compact sets. From our point of view it will be important only that these sets will be Borel subsets of the Cantor set. These subsets will be denoted by the symbols $R_\gamma(n, x)$ where $\gamma$ belongs to $\{-1\} \cup \omega_1$, $x$ is an element of $X^n \mid \alpha_{\gamma+1}$ and $n$ is a natural number. We shall say that the point $(m, x')$, where $m \in D^\omega$ and $x' \in X_m$, is consistent with the restriction $R_\gamma(n, x)$ or that $(m, x')$ satisfies the condition $R_\gamma(n, x)$, if $(m, x')$ belongs to $\{l_\beta : \beta \leqslant \gamma\}$ or if $m \in R_\gamma(n, x)$ whenever $(x' \mid n) \mid \alpha_{\gamma+1} = x$. The set $R_\gamma(n, x)$ will be defined in the $\gamma$-step of induction. The points of $L$ will be defined in such a way that they will be consistent with the restrictions.

If $x \in X^n \mid \alpha_\gamma$, where $\gamma < \omega_1$, and $\lambda \leqslant \gamma$ then put

$$i_\lambda(x) = \begin{cases} \inf\left\{0 \leqslant j < n : x(j) \in \left\{a^0_{\alpha_\lambda}, a^1_{\alpha_\lambda}\right\}\right\} \\ \qquad\qquad \text{if } \left\{0 \leqslant j < n : x(j) \in \left\{a^0_{\alpha_\lambda}, a^1_{\alpha_\lambda}\right\}\right\} \neq \varnothing, \\ -1 \quad \text{otherwise.} \end{cases}$$

We shall split our consideration into three parts. At first we are going to formulate, without proving anything, some conditions which our restrictions are supposed to satisfy. After that we shall prove that the restrictions will ensure that $L$ will not have points of condensation. At the end of this note we shall show how to define the consecutive points of $L$.

*First step of the induction.* Write $E = X \mid \alpha_0 \setminus \{a^0_{\alpha_0}, a^1_{\alpha_0}\}$, and let $w$ be a function $w$: $E \to P_1$ from $E$ onto $P_1$ such that for every $p \in P_1$, the set $w^{-1}(p)$ is infinite and for every $x \in E$, $w(x) \in N_{r(x)}$. Let us assume that

$$(1_{r,-1}) \qquad R_{-1}(1, x) = \begin{cases} Z(w(x)), & \text{if } x \in E, \\ B(x), & \text{if } x \in \left\{a^0_{\alpha_0}, a^1_{\alpha_0}\right\}, \end{cases}$$

and

$$(2_{r,-1}) \qquad I_{-1}(x) = \begin{cases} w(x), & \text{if } x \in E, \\ \varnothing, & \text{if } x \in \left\{a^0_{\alpha_0}, a^1_{\alpha_0}\right\}. \end{cases}$$

If $n > 1$ and $x \in X^n \mid \alpha_0$ then one can define $R_{-1}(n, x)$ and $I_{-1}(x) = (I_{-1,0}(x), \ldots, I_{-1,n-1}(x))$ in such a way that the following conditions hold:

$(3_{r,-1})$ For $x \in X^n \mid \alpha_0$ and $n \geqslant 2$, $I_{-1}(x)$ is an extension of $I_{-1}(x \mid n-1)$.

$(4_{r,-1})$ Let us assume that $j < n-1$, $x \in X^n \mid \alpha_0$, $I_{-1,j}(x) = p$, where $p \in P_{j'}$. Then $I_{-1,j+1}(x)$ is an extension of $p$ belonging to $P_{j'+1}$, if $R_{-1}(j+1, x \mid j+1)$ is not empty and $x(j')$ is not an element of $\{a^0_{\alpha_0}, a^1_{\alpha_0}\}$; and $I_{-1,j+1}(x) = \varnothing$ otherwise.

$(5_{r,-1})$ Let us assume that $x \in X^n \mid \alpha_0$ and for every $j < n-1$, $I_{-1,j}(x) = \varnothing$. Then $I_{-1,n-1}(x)$ is an element of $P_1$ if $R_{-1}(n-1, x \mid n-1)$ is not empty and $x(0)$ is not an element of $\{a^0_{\alpha_0}, a^1_{\alpha_0}\}$; and $I_{-1,n-1}(x) = \varnothing$ otherwise.

($6_{r,-1}$) If $x \in X^n \mid \alpha_0$ then

$$R_{-1}(n, x) = \begin{cases} R_{-1}(n-1, x \mid n-1) \cap B(x) \cap Z(p), & \text{if } p = I_{-1,n-1}(x), \\ R_{-1}(n-1, x \mid n-1) \cap B(x), & \text{if } -1 < i_0(x) < n, \\ \varnothing, & \text{otherwise.} \end{cases}$$

($7_{r,-1}$) For every $x \in X^n \mid \alpha_0$ such that $I_{-1}(x) \neq (\varnothing, \dots, \varnothing)$, $R_{-1}(n, x)$ is a subset of $\bigcap \{Z(p) : \text{there is } i < n \text{ such that } I_{-1,i}(x) = p\}$.

($8_{r,-1}$) $R_{-1}(n, x)$ is compact; from our point of view it is important only that $R_{-1}(n, x)$ is a Borel subset of the Cantor set.

($9_{r,-1}$) Let $n$ be greater than 1, $i < n$, $x$ an element of $X^{n-1} \mid \alpha_0$, $S_i(x) = \{y \in X^n \mid \alpha_0 : x(j) = y(j)$, for $j < i$, and $y(j) = x(j-1)$, for $j > i\}$. If it is consistent with ($4_{r,-1}$) and ($5_{r,-1}$) to assume that there is $y_0 \in S_i(x)$ and $p \in P_n$ such that $I_{-1,n-1}(y_0) = p$, then the set $\{y \in S_i(x) : I_{-1,n-1}(y) = p \text{ and } r(y) = r(y_0)\}$ is infinite.

If $\gamma \in \omega_1$ then one can define sets $R_\gamma(n, x)$, $I_\gamma(x)$, for $n \in N$ and $x \in X^n \mid \alpha_{\gamma+1}$ in such a way that the following conditions hold:

($1_{r,\gamma}$) If $x \in X^n \mid \alpha_{\gamma+1}$ and $i_\gamma(x) = -1$ or $i_\gamma(x) = n-1$, then $R_\gamma(n, x) = R_{\gamma'}(n, x \mid \alpha_{\gamma'+1})$, where

$$\gamma' = \begin{cases} \sup\{\lambda < \gamma : -1 < i_\lambda(x) < n\}, & \text{if } \{\lambda < \gamma : -1 < i_\lambda(x) < n\} \neq \varnothing, \\ -1 & \text{otherwise,} \end{cases}$$

and $I_\gamma(x) = (\varnothing, \dots, \varnothing)$.

($2_{r,\gamma}$) For every $x$ in $X^n \mid \alpha_{\gamma+1}$ and $n > 2$, $I_\gamma(x)$ is an extension of $I_\gamma(x \mid n-1)$.

($3_{r,\gamma}$) If $x \in X^n \mid \alpha_{\gamma+1}$ and there exists $t \in N$ such that $0 < t < n$ and $\{x(t-1), x(t)\}$ is a subset of $\{a^k_{\alpha_\lambda} : k \in D, \lambda \leqslant \gamma + 1\}$, then $I_{\gamma,n-1}(x) = \varnothing$.

($4_{r,\gamma}$) Suppose that $x$ is an element of $X^n \mid \alpha_{\gamma+1}$, $-1 < i_\gamma(x) < n-1$, $\{x(t-1), x(t)\}$ is not a subset of $\{a^k_{\alpha_\lambda} : k \in D, \lambda \leqslant \gamma + 1\}$, for $0 < t < n$, and

$$I_\gamma(x \mid n-1) = (\varnothing, \dots, \varnothing).$$

Then $I_{\gamma,n-1}(x)$ is an element of $P_1$, if $R_\gamma(n-1, x \mid n-1)$ is not empty, $x(0)^{-1}(1)$ is not a subset of $\alpha_\gamma$ and $x(0)$ does not belong to $\{a^0_{\alpha_{\gamma+1}}, a^1_{\alpha_{\gamma+1}}\}$; if $x(0)^{-1}(1) \subset \alpha_\gamma$ then

$$I_{\gamma,n-1}(x) = \{I_{\gamma'',0}(x \mid \alpha_{\gamma''+1}), \dots, I_{\gamma'',n-1}(x \mid \alpha_{\gamma''+1})\} \cap P_1,$$

where $\gamma'' = \sup\{\lambda < \gamma : -1 < i_\lambda(x) < i_\gamma(x)\}$, if $\{\lambda < \gamma: -1 < i_\lambda(x) < i_\gamma(x)\}$ is not empty and $\gamma'' = -1$ otherwise; $I_{\gamma,n-1}(x) = \varnothing$, if none of the above cases hold.

($5_{r,\gamma}$) Suppose that $x$ is an element of $X^n \mid \alpha_{\gamma+1}$, $\{x(t-1), x(t)\}$ is not a subset of $\{a^k_{\alpha_\lambda} : k \in D, \lambda \leqslant \gamma + 1\}$, for $0 < t < n$,

$$I_{\gamma,j}(x) = p$$

where $p$ is an element of $P_{j'}$ and $j < n-1$. Then $I_{\gamma,j+1}(x)$ is an extension of $p$ belonging to $P_{j'+1}$, if $R_\gamma(j+1, x \mid j+1)$ is not empty, $x(j')$ is not an element of $\{a^0_{\alpha_{\gamma+1}}, a^1_{\alpha_{\gamma+1}}\}$ and it is not true that, for every $k \leqslant j'$, $x(k)^{-1}(1)$ is a subset of $\alpha_\gamma$ and $i_\gamma(x) > j'$; if $x(k)^{-1}(1) \subset \alpha_\gamma$, for $k \leqslant j'$ and $i_\gamma(x) > j'$ then

$$I_{\gamma,j+1}(x) = \{I_{\gamma'',0}(x \mid \alpha_{\gamma''+1}), \dots, I_{\gamma'',j+1}(x \mid \alpha_{\gamma''+1})\} \cap P_{j'+1},$$

where $\gamma''$ is defined in the same way as in $(4_{r,\gamma})$; $I_{\gamma,j+1}(x) = \varnothing$, if none of the above cases hold.

$(6_{r,\gamma})$ Let $x$ be an element of $X^n \mid \alpha_{\gamma+1}$. Then

$$R_\gamma(n, x) = \begin{cases} \varnothing, & \text{if } \{x(n-2), x(n-1)\} \subset \{a^k_{\alpha_\lambda} : k \in D, \lambda \leqslant \gamma + 1\}, \\ R_\gamma(n-1, x \mid n-1) \cap R_{\gamma'}(n, x \mid \alpha_{\gamma'+1}) \cap Z(p), & \text{if } p = I_{\gamma,n-1}(x), \\ R_\gamma(n-1, x \mid n-1) \cap R_{\gamma'}(n, x \mid \alpha_{\gamma'+1}), & \\ & \text{if none of the above cases hold,} \end{cases}$$

where $\gamma'$ is defined in the same way as in $(1_{r,\gamma})$.

$(7_{r,\gamma})$ The set $R_\gamma(n, x)$ is a subset of $\bigcap \{Z(p) : \text{there is } i < n \text{ such that } I_{\gamma,i}(x) = p\}$ for $x \in X^n \mid \alpha_{\gamma+1}$, if $I_\gamma(x) \neq (\varnothing, \ldots, \varnothing)$.

$(8_{r,\gamma})$ $R_\gamma(n, x)$ is a compact subset of the Cantor set for $x \in X^n \mid \alpha_{\gamma+1}$.

$(9_{r,\gamma})$ If $2 \leqslant n$, $i < n$, $x \in X^{n-1} \mid \alpha_{\gamma+1}$, $0 \leqslant i_\gamma(x) \leqslant i - 1$ then put $S_i(x) = \{y \in X^n \mid \alpha_{\gamma+1} : y(j) = x(j)$ for $j < 1$, $y(j) = x(j-1)$ for $i < j \leqslant n-1$ and the set $y(i)^{-1}(1)$ is finite$\}$. If it is consistent with $(3_{r,\gamma})$, $(4_{r,\gamma})$, and $(5_{r,\gamma})$ to assume that there is $y_0 \in S_i(x)$ such that $I_{\gamma,n-1}(y_0) = p$, for $j \in N$ and $p \in P_j$, then the set $\{y \in S_i(x) : I_{\gamma,n-1}(y) = p$ and $y \mid \alpha_\gamma = y_0 \mid \alpha_\gamma\}$ is infinite.

Let us assume that $0 \leqslant \beta < \omega_1$, $y = (y(n))_{n=1}^\infty$ is an element of $X^\omega$ such that $y(n)$ is not an element of $\{a^0_{\alpha_\beta}, a^1_{\alpha_\beta}\}$ and $m$ belongs to $M$. We shall show that the restrictions $R_\gamma(n, x)$, for $x \in X^n \mid \alpha_{\gamma+1}$, $n \in N$ and $\gamma < \beta$, will ensure that $(m, y)$ will not be a point of condensation of $L$. If there is a natural number $n$ and $\gamma < \beta$ such that $R_\gamma(n, (y \mid \alpha_{\gamma+1}) \mid n) = \varnothing$ then according to the condition attached to $R_\gamma(n, (y \mid \alpha_{\gamma+1}) \mid n)$ the set

$$U = M \times \{y' \in X^\omega : (y \mid \alpha_{\gamma+1}) \mid n = (y' \mid \alpha_{\gamma+1}) \mid n\}$$

will have at most a countable intersection with $L$. Let us assume that for every natural number $n$ and $\gamma < \beta$ the set $R_\gamma(n, (y \mid \alpha_{\gamma+1}) \mid n)$ is not empty. Then one can show that there exist sequences $(p_n)_{n=1}^\infty$, $(\gamma_n)_{n=1}^\infty$ and $(s_n)_{n=1}^\infty$ such that $p_n \in P_n$, $p_{n+1} \mid n = p_n$, $\gamma_n \in \{-1\} \cup \beta$, $s_n$ is a natural number, $s_n < s_{n+1}$ and $R_{\gamma_n}(s_n, (y \mid \alpha_{\gamma_{n+1}}) \mid s_n)$ is a subset of $Z(p_n) \cap B(y \mid s_n)$. Let $p$ be an element of $(N_0 \oplus N_1)^\omega$ such that $p \mid n = p_n$ for every natural $n$. The family $\{Z(p \mid n) \cap B(y \mid s_n) : n \in N\}$ consists of the decreasing sequence of compact nonempty subsets, so

$$\bigcap \{Z(p \mid n) : n \in N\} = f(p) \in B,$$

by Lemma 4. Then $\bigcap \{B(y \mid s_n) : n \in N\} = (r(y(0)), r(y(1)), \ldots) = f(p) \in B$. From the last fact it follows that $(m, y)$ will not be a point of condensation of $L$ because elements of $L$ are chosen from the closed subset $P = \bigcup \{\{m\} \times X_m : m \in M\}$ of $M \times X^\omega = (D^\omega \setminus B) \times X^\omega$ and $(m, y)$ does not belong to $P$.

We shall show that there are $m_\beta \in M \setminus \{m_\gamma : \gamma \in \{-1\} \cup \beta\}$ and $x_\beta \in X_{m_\beta}$ such that $l_\beta = (m_\beta, x_\beta)$ is consistent with the defined restrictions.

If $\beta = -1$, then it is enough to put $l_{-1} = (m_{-1}, x_{-1})$, where $m_{-1}$ is an arbitrary element of $M$ and $x_{-1}$ an element of $X_{m_{-1}}$. Let us assume that $\beta \geqslant 0$. In the sequel we shall need the following lemma.

LEMMA 5. *Suppose* $\beta > 0$ *and* $y = (y(0), y(1)) \in X^2 \mid \alpha_0$, *where* $y(1) \in \{a^0_{\alpha_\beta}, a^1_{\alpha_\beta}\} \mid \alpha_0$ *and* $I_{-1,1}(y) = p \in P_2 \cap (N_{r(y(0))} \times N_{r(y(1))})$. *Then there is a Borel subset* $C(y)$ *of the Cantor set such that* $H(p) \subset C(y)$, *and for every* $m$ *in* $C(y)$ *there is* $x$ *in* $X_m$ *satisfying the following conditions*: $(x \mid \alpha_0) \mid 2 = y$ *and* $(m, x)$ *is consistent with the restrictions defined in the* $\gamma$*-step of the induction, for every* $\gamma < \beta$.

The proof of Lemma 6 will appear in **[AL$_3$]**. It is much more difficult than anything what we were supposed to prove before. In the proof of Lemma 6 we use Lemma 3 and $(9_{r,\gamma})$ for $\gamma \in \{-1\} \cup \beta$.

If $\beta = 0$ then we do not need Lemma 6. Indeed, if $\beta = 0$ then let $m_0 = (m_0(n))_{n=0}^{\infty}$ be an element of $M \setminus \{m_{-1}\}$ and $x_0$ an element of $X_{m_0}$ such that $x_0(0) \in \{a^0_{\alpha_0}, a^1_{\alpha_0}\}$. The point $l_0 = (m_0, x_0)$ is consistent with the restrictions defined in the $-1$-step of the induction because $R_{-1}(n, (x_0 \mid n) \mid \alpha_0) = B(x_0 \mid n)$, for $n \in N$. Let us assume that $\beta > 0$ and $p = (p(0), p(1))$ be an element of $P_2$. By the definition of $w$ there is $x' \in E$ such that $w(x') = I_{-1}(x') = p(0)$ and $p(0) \in N_{r(x')}$. Put $y_0 = (x', a^k_{\alpha_\beta} \mid \alpha_0)$, where $p(1) \in N_k$. By $(9_{r,-1})$, for $n = 2$, $i = 0$, $x = a^k_{\alpha_\beta} \mid \alpha_0$ and $y_0 = (x', a^k_{\alpha_\beta} \mid \alpha_0)$, there is $y_p \in S_i(x)$ such that $r(y_0) = r(y_p)$ and $I_{-1,1}(y_p) = p$. By Lemma 5 there exists a Borel subset $C(y_p)$ of the Cantor set which contains $H(p)$. Notice that the set $K = \bigcup \{C(y_p) : p \in P_2\}$ is a Borel set, as a countable union of Borel sets, and it contains $B$ because $\bigcup \{H(p) : p \in P_2\} = B$. Thus $(M \setminus \{m_\gamma : \gamma < \beta\}) \cap K \neq \varnothing$ because $M$ is not a Borel set. From the last fact it follows that there exists $p$ in $P_2$ such that $W = (M \setminus \{m_\gamma : \gamma < \beta\}) \cap C(y_p)$ is not empty. Let $m_\beta$ be an element of $W$. By Lemma 5 we infer that there exists $x_\beta$ in $X_{m_\beta}$ such that $l_\beta = (m_\beta, x_\beta)$ is consistent with the restrictions defined in the $\gamma$-step of the induction for $\gamma < \beta$.

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